The open functor from MT-algebras to frames has no adjoint

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Abstract. We study epis and monos in the category of MT-algebras and use them to justify that the open functor from the category of McKinsey-Tarski algebras to the category of frames does neither possess a left or right adjoint.

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- Show equalizers exist through subalgebra structure on boolean algebra and factorization (epi, mono) through image and relativization. Coequalizers existence in terms of relativizations.
- Explicitly describe epis and monos, extremal epis and monos and factorization becomes extremal epis and mono, also check when is it balanced.
- Use epis to show $\mathcal{O}$ does not have right adjoint and similar reason cannot be done for left adjoint. Showing $\mathcal{O}$ has no left adjoint through CBA.

1. Introduction

Plan for intro:

- Pointfree topology. Two approaches: frames and and MT-algebras. There is a natural connection between them. The open element functor, which is in a sense a forgetful functor; it simply forgets the ambient boolean algebra and only remembers the frame of open elements.
- The Funayama envelope is a tool that shows that the open set functor is essentially surjective. One might hope that the Funayama envelope is adjoint to the open element functor, but it is not even clear in what way it can be extended to a good functor because frame maps dont always extend to complete boolean maps on their Funayama envelope, and consequently

this assignment on objects can not be extended to an adjoint to the open element functor.
- This raises the question whether the open element functor has an adjoint at all. In this paper we show that this is not the case. There is no left or right adjoint to the open element functor.
- The underlying reason is that there is a crucial difference between morphisms in MT and morphisms in frames, a behavior which is exhibited by the character of epis and monos in the respective categories.
- Epis and monos, and their extremal counterparts, are well understood in topological spaces. Besides, every extremal mono/epi is regular in topological spaces.
- The hierarachy of epis and monos in the category of frames is more delicate. Extremal epis are still regular and are precisely the surjective frame maps. Non-extremal epimorphisms are quite difficult to understand. On the side of monomorphisms, the good part is that they are injective. The bad part is that understanding extremal ones becomes difficult. Extremal monos are not regular.
- In this paper we characterize epis and monos in the category of MT-algebras. We show that extremal epis/monos are regular. In this sense MT-algebras are more faithful to spaces. To do this, we show that the equalizers and coequalizers in CBA can be equipped with interior operators, yielding the likewise categorical constructions in MT-algebras. We obtain that MT is a complete category as a corollary.

2. The category of complete boolean algebras

In this section, we recall various properties of the category of complete boolean algebras. Most, if not all, results are known and folklore. Our main reference for categorical notions is [AHS06].

Let CBA be the category of complete boolean algebras and complete boolean homomorphisms. It is straightforward to verify that the category CBA has all products and equalizers. Products are simply the cartesian products with the operations computed pointwise. If $f, g : A \to B$ are complete boolean homomorphisms, then their equalizer is

$$E = \{a \in A \mid f(a) = g(a)\},$$

viewed as a boolean subalgebra of A. It follows that E is a complete subalgebra since f and g are complete homomorphisms. Since every category with products and equalizers is complete (see, e.g., [AHS06, Thm. 12.3]), we obtain the following:

Theorem 2.1. CBA *is complete.*

We next observe the existence of coequalizers in CBA. These are easiest to describe in terms of *relativizations* [RS63, Sec. II.6 and III.2]. Given a complete boolean algebra A and an element $a \in A$. The *relativization* of A to a is the principal ideal $A_a = \{b \in A \mid b \leq a\}$ with the order inherited from A. Then A_a is a complete boolean algebra, where $\neg_a b = \neg b \wedge a$ for $b \in A_a$, and the *relativization map* $A \to A_a$ defined by $b \mapsto b \wedge a$ is a surjective complete boolean homomorphism.

Remark 2.2. It is will known that principal ideals of complete boolean algebras play a similar role as ideals for boolean algebras. That is, for each complete boolean homomorphism $f : A \to B$, the set $f^{-1}(0)$ is closed under arbitrary joins, and hence

a principal ideal. Consequently, up to isomorphism all surjective complete boolean homomorphisms with domain A correspond to some relativization of A.

We now describe the connection between relativizations and coequalizers. Let $f, g : A \to B$ be complete boolean homomorphisms and put

$$d = \bigwedge_{a \in A} \Big(f(a) \leftrightarrow g(a) \Big),$$

where $b \leftrightarrow c := (\neg b \vee c) \wedge (\neg c \vee b)$. Then the relativization from B to d is the coequalizer of f and g.

Proposition 2.3. CBA *has coequalizers*

Moreover, every relativization is a coequalizer. Indeed, let A be a complete boolean algebra, $a \in A$, and suppose $f : A \to A_a$ is the relativization map. Then its kernel

$$\ker(f) = \{(a, b) \in A \times A \mid f(a) = f(b)\} = \{(b, c) \in A \times A \mid b \wedge a = c \wedge a\}$$

is a complete boolean algebra and we consider the *kernel pair* $\pi_1, \pi_2 : \ker(f) \to A$ of the corresponding projections. (Alternatively, the kernel pair can be thought of as the pullback of f with itself.) Then the coequalizer of π_1 and π_2 is the relativization of A to

$$d = \bigwedge_{(b,c) \in \ker(f)} (b \leftrightarrow c),$$

but $(b, c) \in \ker(f)$ iff $b \wedge a = c \wedge a$ iff $a \leq b \leftrightarrow c$. Therefore, $d = a$, so A_a is the coequalizer of the kernel pair of the corresponding relativization map. By Remark 2.2, we can conclude that all surjective complete boolean homomorphisms are the coequalizer of their kernel pair, and hence regular epimorphisms.

Proposition 2.4. *All surjective maps in* CBA *are regular epimorphisms.*

We point out that CBA does not have all coproducts, and hence is not cocomplete (see, e.g., [AHS06, p. 186]). In particular, the existence of pushouts is not automatic. Nonetheless, CBA enjoys the *strong amalgamation property*, that is, whenever there are injective complete boolean homomorphisms $f : A \to B$ and $g : A \to C$, then there exist injective complete boolean homomorphisms $f' : B \to D$ and $g' : C \to D$ such that $f' \circ f = g' \circ g$ and $f'[B] \cap g'[C] = (f' \circ f)[A]$.

Theorem 2.5 ([Lag74])**.** CBA *has the strong amalgamation property.*

LaGrange used this result to show that all epimorphisms in CBA are surjective. Since all surjective maps are regular epimorphisms by Proposition 2.4, it follows that all epimorphisms are regular.

Corollary 2.6. *In* CBA*, epimorphisms are surjective and extremal.*

Additionally, Lagrange's Theorem can be used to prove the analogous statement about monomorphisms. While a simple argument shows that monomorphisms are injective, we need to use the strong amalgamation property to realize an injective map as an equalizer. Indeed, if $f : A \to B$ is injective, then the strong amalgamation property yields a injective morphisms $g, h : B \to C$ such that $g \circ f = h \circ f$ and $g[B] \cap h[B] = (g \circ f)[A]$. It follows that $f[A] = \{b \in B \mid g(b) = h(b)\}$, so $A \cong f[A]$ is the equalizer since f is injective.

Corollary 2.7. *In* CBA*, monomorphisms are injective and regular.*

Every morphism $f : A \to B$ factors as

$$A \xrightarrow{e} f[A] \xrightarrow{m} B,$$

where e is the surjective complete homomorphism onto the image and m is the injective inclusion of the image into B. Moreover, since every epimorphism is regular and every monomorphism is regular, CBA is simultaneously (Extremal Epi, Mono)-structured and (Epi, Extremal Mono)-structured.

Corollary 2.8. CBA *has* (Regular Epimorphisms, Regular Monomorphisms)*-factorization.*

Since monomorphisms in CBA are injective, the subobjects of $B \in$ CBA correspond to complete subalgebras, and since epimorphisms are surjective, every quotient (in the sense of being the codomain of an epimorphism) of B corresponds to a relativization, and hence an element of B. Consequently:

Corollary 2.9. *The category* CBA *is wellpowered and co-wellpowered.*

To conclude this section we review the position of CBA as a subcategory of the category of *frames*. We recall that a complete lattice L is a *frame* if it satisfies the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

A *frame morphism* is a map between frames preserving arbitrary joins and finite meets. We let Frm denote the category of frames and frame morphisms.

Every complete boolean algebra, is a frame, and moreover frame homomorphisms between complete boolean algebras are complete boolean homomorphisms. Thus, CBA is a full subcategory subcategory of Frm (see, e.g, [Joh82, p. 56]). In contrast to CBA, the category Frm is both complete and cocomplete [PP12, p. 59]. Consequently, CBA is not a reflective subcategory of Frm (see also [Joh82, p. 57]).

3. Preliminaries

Discuss the adjunction between CBA and Set and how that can be extended to MT and Top. Recall that the forgetful functor from Top to set is a topological functor. That Set is both reflective and coreflective subcategory of Top (discrete and indiscrete). Similarly, we can think of CBA as a coreflective or reflective subcategory of MT. Moreover, we can show that the reflection (coreflection) morphisms are identity morphisms in CBA. It follows that the forgotful functor from MT to CBA is topological. Consequently, there are categrocial descriptions of extremal monos and extremal epis in MT in terms of CBA, and several such results follow from, e.g., Joy of Cats.

In this section, we recall the definitions of frames, boolean frames (complete boolean algebras) and MT-algebras and some of the categorical properties of category of frames (Frm) and complete boolean algebras (CBA) and the categorical connection between Frm and CBA which will be easier to compare it to the categorical properties of MT-algebras in the rest of the sections. **Not everything mentioned in this paragraph is discussed below. In particular, categorical properties are not really mentioned. Also the relation between CBA and MT should be discussed. That the former is both reflective and coreflective in the latter.**

We recall that a complete lattice L is a *frame* if it satisfies the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\},$$

. A *frame morphism* is a map between frames preserving arbitrary joins and finite meets. We let Frm denote the category of frames and frame morphisms.

If the complete lattice L is boolean, it is a boolean frame, since negation gives the meet-infinte distributive law too, boolean frames are nothing but complete boolean algebras and moreover frame homomorphisms between complete boolean algebras are complete boolean homomorphisms. Thus, we can let CBA be a full subcategory subcategory of Frm consisting of complete boolean algebras with complete boolean homomorphisms.

While a standard example of a frame is $\Omega(X)$ lattice of open sets of a topological space X, a boolean frame is $\mathcal{P}(X)$ lattice of all subsets of a topological space X and the fixed points of interior operator on X gives $\Omega(X)$. But $\mathcal{P}(X)$ is a spatial boolean frame, there exists non spatial boolean frame. McKinsey and Tarski [MT44] (see also [Nöb54]) generalized closure/interior operators on powerset algebras to arbitrary boolean algebras. This led to the theory of McKinsey-Tarski algebras, which provides an alternative (and more expressive) pointfree approach to topology; see [BR23].

We recall that an *interior operator* on a bounded lattice L is a unary function $\Box : L \to L$ satisfying Kuratowski's axioms for all $a, b \in L$:

$$\Box 1 = 1, \quad \Box(a \wedge b) = \Box a \wedge \Box b, \quad \Box a \leq a, \quad \text{and} \quad \Box a \leq \Box\Box a.$$

Definition 3.1. Let B be a boolean algebra and $\Box$, an interior operator on it.

(1) $\boldsymbol{B} = (B, \Box)$ is called as an interior algebra
(2) An element $a \in B$ is called *open* if $a = \Box a$.
(3) Let $\mathcal{O}(\boldsymbol{B})$ be the collection of open elements of $\boldsymbol{B}$.

Definition 3.2.

(1) A *McKinsey-Tarski algebra* or simply an *MT-algebra* is an interior algebra $\boldsymbol{M} = (M, \Box)$ where M is a complete boolean algebra.
(2) An *MT-morphism* between MT-algebras $\boldsymbol{M}$ and $\boldsymbol{N}$ is a complete boolean morphism $f : M \to N$ such that $f(\Box a) \leq \Box f(a)$ for each $a \in M$.
(3) Let MT be the category of MT-algebras and MT-morphisms.

For a topological space X, the pair $(\mathcal{P}(X), \mathrm{int})$ is an MT-algebra and if $f : X \to Y$ is a continuous map between X and Y, then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is an MT-morphism. So we have the following theorem:

Theorem 3.3. [BR23, Thm. 3.6] $\mathcal{P} : \mathsf{Top} \to \mathsf{MT}^{\mathrm{op}}$ *is a left adjoint functor.*

For the other way, for an MT-algebra $\boldsymbol{M}$, let $at(\boldsymbol{M})$ be the set of atoms of M. For $a \in M$, put $\eta(a) = \{x \in at(\boldsymbol{M}) \mid x \leq a\}$. The collection $\eta[\mathcal{O}(\boldsymbol{M})]$ is a topology on $at(\boldsymbol{M})$. If $h : \boldsymbol{M} \to \boldsymbol{N}$ is an MT-morphism, then the left adjoint of h restricts to a continuous map $at(h) : at(\boldsymbol{N}) \to at(\boldsymbol{M})$. Thus, we get the following theorem:

Theorem 3.4. [BR23, Thm. 3.15] $at : \mathsf{MT}^{\mathrm{op}} \to \mathsf{Top}$ *is a right adjoint to* $\mathcal{P}$.

We call $\boldsymbol{M}$ *spatial* if M is atomic. Let SMT be the full subcategory of MT consisting of spatial MT-algebras. For each $\boldsymbol{M} \in \mathsf{SMT}$, $\eta_M : \boldsymbol{M} \to \mathcal{P}(at(\boldsymbol{M}))$ is an isomorphism of MT-algebras, that is structure preserving bijections. For

each $X \in \mathsf{Top}$, let $\varepsilon_X : X \to at(\mathcal{P}X)$ be given by $\varepsilon_X(x) = \{x\}$. Then ε_X is a homeomorphism and we have:

Theorem 3.5 ([BR23, Thm. 3.22])**.** *$(\mathcal{P}, at)$ is an adjunction between Top and $\mathsf{MT}^{\mathrm{op}}$ whose unit is given by $\varepsilon : 1_{\mathsf{Top}} \to at \circ \mathcal{P}$ and counit by $\eta : \mathcal{P} \circ at \to 1_{\mathsf{MT}}$. This adjunction restricts to an equivalence between Top and $\mathsf{SMT}^{\mathrm{op}}$.*

There is also a close connection between MT-algebras and frames. For each MT-algebra M, let $\mathcal{O}M$ be the fixpoints of $\Box$, which we call *open elements*. $\mathcal{O}M$ is a subframe of M, hence $\mathcal{O}M$ is a frame. Moreover, if $f : M \to M'$ is an MT-morphism, then its restriction $f|_{\mathcal{O}M} : \mathcal{O}M \to \mathcal{O}M'$ is a frame morphism. Thus, we have the following theorem:

Theorem 3.6 ([BR23, Thm. 4.2])**.** *$\mathcal{O} : \mathsf{MT} \to \mathsf{Frm}$ is an essentially surjective functor.*

In particular, for each frame L, there is an MT-algebra M such that $L \cong \mathcal{O}M$ which can be constructed by taking the *Funayama envelope* $\mathcal{F}L$ of L [Fun59]. $\mathcal{F}L$ can be realized as the MacNeille completion of the boolean envelope of L [Grä78, Sec. II.4]. However, the assignment $L \mapsto \mathcal{F}L$ cannot be extended to a functor $\mathsf{Frm} \to \mathsf{MT}$ with $f = \mathcal{O}\mathcal{F}f$ for every frame morphism f. Indeed, there are frame morphisms that do not lift to complete boolean homomorphisms between their Funayama envelopes (see e.g., [BR23, Example 4.4]).

4. Equalizers and coequalizers in the category of MT-algebras

In this section we,

Mention what we do in this section.

In the category CBA of complete boolean algebras, the equalizer of two morphisms $f, g : A \to C$ is simply the complete boolean subalgebra $B = \{a \in A \mid f(a) = g(a)\}$ of A with the inclusion map. We show that this same construction lifts to MT once we equip the subalgebra with a suitable interior operator.

Compare to quotient topology. And mention that the embedding is extremal mono at some point.

Lemma 4.1. *If $B \subseteq M$ is a complete boolean subalgebra of an MT-algebra $\boldsymbol{M} = (M, \Box)$, then*

$$\Box_B b = \bigvee \{u \in B \cap \mathcal{O}(\boldsymbol{M}) \mid u \leq b\}$$

defines an interior operator on B with $\mathcal{O}(\boldsymbol{B}) = B \cap \mathcal{O}(\boldsymbol{M})$, where $\boldsymbol{B} = (B, \Box_B)$.

Proof. Let $b, b' \in B$. Clearly, $\Box_B b \leq b$ and $\Box_B b \wedge \Box_B b' = \Box_B(b \wedge b')$. Since $1 \in B \cap \mathcal{O}(\boldsymbol{M})$, $\Box_B 1 = 1$. Also, if $u \in B \cap \mathcal{O}(\boldsymbol{M})$ and $u \leq b$, then $u \leq \Box_B b$. Therefore, $\Box_B b \leq \Box\Box_B b$, so $\Box_B$ is an interior operator. Thus, $\Box\Box_B b = \Box_B b$ and hence $\mathcal{O}(\boldsymbol{B}) = B \cap \mathcal{O}(\boldsymbol{M})$. Consequently, $\Box_B \Box_B b = \Box_B b$ since $\Box_B b \in B \cap \mathcal{O}(\boldsymbol{M})$. □

Lemma 4.2. *The embedding $e : \boldsymbol{B} \to \boldsymbol{M}$ is an MT-morphism.*

Proof. For $b \in B$, we have $e(\Box_B b) = \Box_B b \leq \Box b = \Box e(b)$ since $\Box_B b$ is a join of open elements below b, and $\Box b$ is the join of all such. □

Lemma 4.3. *A complete boolean homomorphism $h : \boldsymbol{M} \to \boldsymbol{N}$ between MT-algebras is an MT-morphism iff $h[\mathcal{O}(\boldsymbol{M})] \subseteq \mathcal{O}(\boldsymbol{N})$.*

Proof. ($\Rightarrow$) If $u \in \mathcal{O}(\boldsymbol{M})$, then $h(u) = h(\Box u) \leq \Box h(u) \leq h(u)$.
($\Leftarrow$) For $a \in M$, we have

$$\begin{aligned} h(\Box a) &= h\big(\bigvee\{u \in \mathcal{O}(\boldsymbol{M}) \mid u \leq a\}\big) \\ &= \bigvee\{h(u) \mid u \in \mathcal{O}(\boldsymbol{M}), u \leq a\} \\ &\leq \bigvee\{v \in \mathcal{O}(\boldsymbol{N}) \mid v \leq h(a)\} \\ &= \Box h(a). \end{aligned}$$

□

Theorem 4.4. *Let $f, g : \boldsymbol{M} \to \boldsymbol{N}$ be MT-morphisms. Let $B = \{a \in M \mid f(a) = g(a)\}$. Then the embedding $e : \boldsymbol{B} \to \boldsymbol{M}$ is the equalizer of f and g.*

Proof. By Lemma 4.1, $\boldsymbol{B}$ is an MT-algebra and the embedding e is an MT-morphism by Lemma 4.2. Clearly, $f \circ e = g \circ e$. Suppose $e' : \mathbf{C} \to \boldsymbol{M}$ is an MT-morphism such that $f \circ e' = g \circ e'$. By definition of B, we have $e'[C] \subseteq B$. Since e' is an MT-morphism, $e'[\mathcal{O}(\mathbf{C})] \subseteq \mathcal{O}(\boldsymbol{M})$. Hence $e'[\mathcal{O}(\mathbf{C})] \subseteq B \cap \mathcal{O}(\boldsymbol{M}) = \mathcal{O}(\boldsymbol{B})$ by Lemma 4.1. Consequently, the map $h : \mathbf{C} \to \boldsymbol{B}$ defined by $h(c) = e'(c)$ is an MT-morphism by Lemma 4.3, and $e \circ h = e'$. For uniqueness, suppose $h' : \mathbf{C} \to \boldsymbol{B}$ is an MT-morphism with $e \circ h' = e'$. Then $e \circ h' = e \circ h$ as maps $C \to M$. Since e is injective, $h' = h$. □

Corollary 4.5. *The category* MT *has all equalizers.*

In fact, since MT has all products **(Probably this should be explained)**, it follows from Corollary 4.5 that all limits in MT exist (see, e.g., [**?**, p. 113]). Thus, we obtain the following:

Corollary 4.6. *The category* MT *is complete.*

The existence of equalizers in MT also implies that every MT-morphism can be factorized through its image. Given $f : \boldsymbol{M} \to \boldsymbol{N}$, let $B = f[M]$ and equip B with the interior operator

$$\Box_B b = \bigvee\{u \in B \cap \mathcal{O}(\boldsymbol{N}) \mid u \leq b\}.$$

By Lemma 4.1, $\boldsymbol{B} = (B, \Box_B)$ is an MT-algebra with $\mathcal{O}(\boldsymbol{B}) = B \cap \mathcal{O}(\boldsymbol{N})$.

Proposition 4.7. *Every MT-morphism $f : \boldsymbol{M} \to \boldsymbol{N}$ factors as*

$$\boldsymbol{M} \xrightarrow{e} \boldsymbol{B} \xrightarrow{m} \boldsymbol{N}$$

where e is an epimorphism, m is a monomorphism, and $B = f[M]$.

Proof. Let $e : \boldsymbol{M} \to \boldsymbol{B}$ be the corestriction of f. Since f is an MT-morphism, $f[\mathcal{O}(\boldsymbol{M})] \subseteq \mathcal{O}(\boldsymbol{N})$. Hence $e[\mathcal{O}(\boldsymbol{M})] = f[\mathcal{O}(\boldsymbol{M})] \subseteq f[M] \cap \mathcal{O}(\boldsymbol{N}) = B \cap \mathcal{O}(\boldsymbol{N}) = \mathcal{O}(\boldsymbol{B})$. By Lemma 4.3, e is an MT-morphism. Since e is surjective, it is an epimorphism by Corollary 5.2. Let $m : \boldsymbol{B} \to \boldsymbol{N}$ be the inclusion. Then $m[\mathcal{O}(\boldsymbol{B})] = \mathcal{O}(\boldsymbol{B}) = B \cap \mathcal{O}(\boldsymbol{N}) \subseteq \mathcal{O}(\boldsymbol{N})$. By Lemma 4.3, m is an MT-morphism. Since m is injective, it is a monomorphism. Clearly $f = m \circ e$. □

The image factorization constructs a factorization through an MT-algebra structure on the image in the codomain. We now describe an alternative factorization through a relativization of the domain.

Recall (see, e.g., [RS63, p. 95]) that the *relativization* $\boldsymbol{M}_a = \{b \in M \mid b \leq a\}$ of an MT-algebra $\boldsymbol{M}$ to an element $a \in M$ is an MT-algebra, where $\mathcal{O}(\boldsymbol{M}_a) = \{u \wedge a \mid u \in \mathcal{O}(\boldsymbol{M})\}$.

Proposition 4.8. *For every MT-morphism $f : \boldsymbol{M} \to \boldsymbol{N}$, there exists a relativization $\boldsymbol{M}_a$, an epimorphism $g_a : \boldsymbol{M} \to \boldsymbol{M}_a$, and a monomorphism $f_a : \boldsymbol{M}_a \to \boldsymbol{N}$ such that $f = f_a \circ g_a$ and this is universal, i.e., for each MT-morphism $g : \boldsymbol{M} \to \boldsymbol{C}$ and monomorphism $m : \boldsymbol{C} \to \boldsymbol{N}$ such that $f = m \circ g$ there is a unique MT-morphism $h : \boldsymbol{M}_a \to \boldsymbol{C}$ such that $f_a = m \circ h$.*

Proof. Let $a = \bigwedge f^{-1}(1)$ and define $g_a(b) = b \wedge a$ and $f_a(b) = f(b)$ for all $b \in M_a$. Let $g_1, g_2 : \boldsymbol{M}_a \to \boldsymbol{C}$ be MT-morphisms such that $g_1 \circ g_a = g_2 \circ g_a$, then for any $x \in M_a$,

$$g_1(x) = g_1(x \wedge a) = g_1(g_a(x)) = g_2(g_a(x)) = g_2(x \wedge a) = g_2(x).$$

Thus, g_a is an epimorphism. Now, to see f_a is mono, it is enough to see it is injective. Consider $b, c \in M_a$ such that $f_a(b) = f_a(c)$, then $1 = f_a(b) \vee \neg f_a(c) = f_a(b \vee \neg c) = f(b \vee \neg c)$. Then, $b \vee \neg c \in f^{-1}(1)$ and hence $a \leq b \vee \neg c$. Hence,

$$a = (b \vee \neg c) \wedge a = (b \wedge a) \vee (a \wedge \neg c) = b \vee \neg_a c.$$

Thus, $b = c$. Now, to see the universal property, let $h : M_a \to C$ be defined as $h(g_a(b)) := g(b)$. It is clearly a well defined MT-morphism, to see it is unique, say there exists an MT-morphism $h' : \boldsymbol{M}_a \to \boldsymbol{C}$, such that $f_a = m \circ h'$. Then

$$m(h'(g_a(b))) = f_a(g_a(b)) = f(b) = m(g(b))$$

for all $b \in A$. Since m is a monomorphism, $h'(g_a(b)) = g(b) = h(g_a(b))$. □

The relativization construction also yields coequalizers. Given MT-morphisms $f, g : \boldsymbol{M} \to \boldsymbol{N}$, define $a = \bigwedge_{x \in M}(f(x) \leftrightarrow g(x))$, where $b \leftrightarrow c = (b \wedge c) \vee (\neg b \wedge \neg c)$ **Is this common notation? I would expect it to be** $(\neg b \vee c) \wedge (\neg c \vee b)$. Let $q_a : \boldsymbol{N} \to \boldsymbol{N}_a$ be the relativization map $q_a(y) = y \wedge a$.

Theorem 4.9. *The map q_a is the coequalizer of f and g.*

Proof. For any $x \in M$, we have $a \leq f(x) \leftrightarrow g(x)$, so $f(x) \wedge a = g(x) \wedge a$. Thus $q_a(f(x)) = q_a(g(x))$. Suppose $q : \boldsymbol{N} \to \boldsymbol{Q}$ satisfies $q \circ f = q \circ g$. Then for all x, $q(f(x)) = q(g(x))$, so $q(f(x) \leftrightarrow g(x)) = 1$. Hence,

$$q(a) = \bigwedge_{x \in M} q(f(x) \leftrightarrow g(x)) = 1.$$

Define $u : (N_a, \Box_{N_a}) \to \boldsymbol{Q}$ by $u(y \wedge a) = q(y)$. This is well defined: if $y_1 \wedge a = y_2 \wedge a$, then $y_1 \leftrightarrow y_2 \geq a$, so $q(y_1 \leftrightarrow y_2) \geq q(a) = 1$, hence $q(y_1) = q(y_2)$. The map u is an MT-morphism by Lemma 4.3 since $q_a[\mathcal{O}(\boldsymbol{N})] \subseteq \mathcal{O}(N_a, \Box_{N_a})$. Clearly $u \circ q_a = q$. □

Corollary 4.10. *The category* MT *has all coequalizers.*

Mention, perhaps in a remark, that it does not follow that the category MT is cocomplete, since it doesnt have coproducts. Cite ABC paper.

5. Epimorphisms in the category of MT-algebras

In this section we describe epimorphisms and extremal epimorphisms in MT and show extremal and regular coincide.

Theorem 5.1 ([Lag74])**.** *In* CBA*, all epimorphisms are surjective.*

Corollary 5.2. *In* MT*, all epimorphisms are surjective.*

Proof. Let $f : A \to B$ be an epimorphism in MT. By Theorem 5.1, it is enough to show that f is an epimorphism in CBA. Let $g_1, g_2 : B \to C$ be complete Boolean homomorphism such that $g_1 \circ f = g_2 \circ f$. Define $\Box$ on C by $\Box(a) = a$. This defines an MT-algebra $\hat{C}$, and it is easily verified $g_1, g_2 : B \to C$ can be viewed as MT-morphisms $g_1, g_2 : B \to \hat{C}$. Thus, $g_1 = g_2$ since f is an epimorphism in MT. We conclude f is an epimorphism in CBA. □

In order to characterize extremal epimorphisms, we need the following lemma:

Lemma 5.3. *Let $f : \mathbf{M} \to \mathbf{N}$ be a bijective MT-morphism. Then f is an isomorphism iff $\mathcal{O}(f)$ is surjective.*

Proof. Suppose f is an isomorphism, then $f(\Box a) = \Box f(a)$ for all $a \in M$ (see e.g., [Rav25, Rem. 3.6]). Let $v \in \mathcal{O}(\mathbf{N})$, then since f is surjective there exists $a \in \mathbf{M}$ such that $f(a) = v$, since f is open $\Box f(a) = f(\Box a) = v$, and $\Box a \in \mathcal{O}(\mathbf{M})$. Thus, $\mathcal{O}(f)$ is surjective. Conversely, suppose $\mathcal{O}(f)$ is surjective, for $a \in \mathbf{M}$, $\Box f(a) \in \mathcal{O}(\mathbf{N})$, then there exists $u \in \mathcal{O}(\mathbf{M})$ such that $f(u) = \Box f(a) \leq f(a)$. Since f is a bijective MT-morphism, $u \leq \Box a$, and hence $f(u) = f(\Box a) \leq f(\Box a)$. □

Theorem 5.4. *Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be an MT-morphism. The following are equivalent:*

(1) *f is an extremal epimorphism.*
(2) *Both f and $\mathcal{O}(f)$ are surjective.*
(3) *The induced map $f_a : \boldsymbol{M}_a \to \boldsymbol{N}$ is an isomorphism.*

Proof. By Corollary 5.2, all epimorphisms in MT are surjective. We use the factorization $f = f_a \circ g_a$ from Proposition 4.8.

(1)⇒(3): If f is extremal, then since $f = f_a \circ g_a$ with f_a mono, f_a is an isomorphism.

(3)⇒(2): If f_a is an isomorphism, then $\mathcal{O}(f_a)$ is an isomorphism. For any $v \in \mathcal{O}(\boldsymbol{N})$, there exists $u' \in \mathcal{O}(\boldsymbol{M}_a)$ with $f_a(u') = v$. Since $\mathcal{O}(g_a)$ is surjective, there exists $u \in \mathcal{O}(\boldsymbol{M})$ with $g_a(u) = u'$. Then $f(u) = f_a(g_a(u)) = v$. Thus $\mathcal{O}(f)$ is surjective.

(2)⇒(1): Suppose $\mathcal{O}(f)$ is surjective and $f = m \circ g$ with m mono. Since f is surjective, m is bijective. By Lemma 5.3, m is an isomorphism iff $\mathcal{O}(m)$ is surjective. From the universal property, $f_a = m \circ h$ for some h, so $\mathcal{O}(f) = \mathcal{O}(m) \circ \mathcal{O}(h)$, hence $\mathcal{O}(m)$ is surjective. Thus m is an isomorphism. □

Theorem 5.5. *Every extremal epimorphism in MT is regular.*

Proof. Let $h : \boldsymbol{M} \to \boldsymbol{N}$ be extremal epic. Let $P = \{(a, b) \in M \times M \mid h(a) = h(b)\}$ be the kernel pair of h with projections $p_1, p_2 : P \to M$.

By Theorem 4.9, the coequalizer of p_1, p_2 is $q_a : \boldsymbol{M} \to \boldsymbol{M}_a$ where $a = \bigwedge_{(x,y)\in P}(x \leftrightarrow y) = \bigwedge h^{-1}(1)$.

Since $h \circ p_1 = h \circ p_2$, there exists a unique $\overline{h} : \boldsymbol{M}_a \to \boldsymbol{N}$ with $\overline{h} \circ q_a = h$. This $\overline{h}$ is exactly the map h_a from the factorization $h = h_a \circ q_a$, hence is injective. Since h is extremal epic, h_a is an isomorphism by Theorem 5.4. Therefore h is a coequalizer. □

Remark 5.6. In Proposition 4.8 g_a is an onto MT-morphism and $\mathcal{O}(g_a)$ is onto from the definition of opens. Thus, by Theorem 5.4 every MT-morphism f can be factorized into extremal epi and mono. Thus, by Theorem 5.5 regular epi and mono.

6. Monomorphisms in the category of MT-algebras

In this section we describe monomorphisms, extremal monomorphisms in MT.

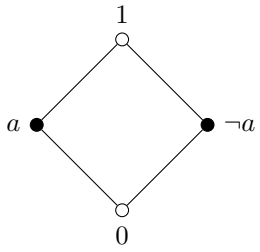


FIGURE 1. The four-element MT-algebra, with open elements shown as unfilled circles

Lemma 6.1. *In* MT*, all monomorphisms are injective.*

Proof. Let $f : B \to C$ be a monomorphism in MT. Suppose $b, b' \in B$ are such that $f(b) = f(b')$. Let A the four element MT-algebra where a denotes some element not equal to 0 or 1, and 0 and 1 are the only opens, see Fig. 1. Pick arbitrary $b_1, b_2 \in b$ and define $g_1, g_2 : A \to B$ by

$$\begin{aligned} g_1(1) &= 1 & g_2(1) &= 1 \\ g_1(a) &= b & g_2(a) &= b' \\ g_1(\neg a) &= \neg b & g_2(\neg a) &= \neg b' \\ g_1(0) &= 0 & g_2(0) &= 0 \end{aligned}$$

Is is straightforward to verify this is an MT-morphism. Moreover, $f \circ g_1 = f \circ g_2$, so $g_1 = g_2$, i.e., $b = g_1(a) = g_2(a) = b'$, so f is injective. □

Theorem 6.2. *Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be an MT-morphism. Let $\boldsymbol{B} = (B, \Box_B)$ where $B = f[M]$ and $\Box_B$ is the induced interior from Lemma 4.1. The following are equivalent:*

1. *f is an extremal monomorphism.*
2. *f is injective and $f[\mathcal{O}(\boldsymbol{M})] = \mathcal{O}(\boldsymbol{N}) \cap B$.*
3. *The corestriction $e : \boldsymbol{M} \to \boldsymbol{B}$ is an isomorphism.*

Proof. Let $f = m \circ e$ be the image factorization with $e : \boldsymbol{M} \to \boldsymbol{B}$ the corestriction and $m : \boldsymbol{B} \to \boldsymbol{N}$ the inclusion.

(1)⇒(3): Immediate from extremality since e is epi.

(3)⇒(2): If e is iso, then f is injective. Also $\mathcal{O}(e)$ is iso, so $f[\mathcal{O}(\boldsymbol{M})] = m[e[\mathcal{O}(\boldsymbol{M})]] = m[\mathcal{O}(\boldsymbol{B})] = \mathcal{O}(\boldsymbol{N}) \cap B$.

(2)⇒(1): Let $f = g \circ e$ with $e : \boldsymbol{M} \to \mathbf{C}$ epi. Then e is surjective. Since f is injective, e is injective, hence bijective. By Lemma 5.3, to show e is an isomorphism, it is enough to show $\mathcal{O}(e)$ is surjective. For $v \in \mathcal{O}(\mathbf{C})$, we have $g(v) \in g[\mathcal{O}(\mathbf{C})] \subseteq \mathcal{O}(\boldsymbol{B})$. Also $g(v) \in g[e[M]] = f[M] = B$. Thus $g(v) \in \mathcal{O}(\boldsymbol{N}) \cap B = f[\mathcal{O}(\boldsymbol{M})] = g[e[\mathcal{O}(\boldsymbol{M})]]$. Since g is injective on $e[M]$ and $v \in e[M]$ (as e is surjective), we get $v \in e[\mathcal{O}(\boldsymbol{M})]$. □

Theorem 6.3. *Every extremal monomorphism in* MT *is regular.*

Proof. Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be an extremal monomorphism. By Theorem 6.2, f is injective and $f[\mathcal{O}(\boldsymbol{M})] = \mathcal{O}(\boldsymbol{N}) \cap f[M]$. Let $\boldsymbol{B} = (B, \Box_B)$ be the MT-algebra structure on $B = f[M]$ given by Lemma 4.1. Then the corestriction $e : \boldsymbol{M} \to \boldsymbol{B}$ is an isomorphism, so we may identify $\boldsymbol{M}$ with $\boldsymbol{B}$ and view f as the inclusion $\boldsymbol{B} \hookrightarrow \boldsymbol{N}$.

Apply Lagrange's strong amalgamation theorem [Lag74] to the inclusion $B \hookrightarrow N$ in CBA. This yields a complete Boolean algebra E and complete embeddings $j_1, j_2 : N \to E$ such that $j_1|_B = j_2|_B$ and $j_1[N] \cap j_2[N] = j_1[B]$.

Let $\mathcal{O}(\mathbf{E})$ be the complete subframe of E generated by $j_1[\mathcal{O}(\boldsymbol{N})] \cup j_2[\mathcal{O}(\boldsymbol{N})]$, and define

$$\Box_E x = \bigvee \{u \in \mathcal{O}(\mathbf{E}) \mid u \leq x\}.$$

Since $\mathcal{O}(\mathbf{E})$ is closed under arbitrary joins and finite meets, this defines an interior operator on E with $\mathcal{O}(\mathbf{E})$ as its set of fixpoints. Thus $\mathbf{E} = (E, \Box_E)$ is an MT-algebra. For $k = 1, 2$, we have $j_k[\mathcal{O}(\boldsymbol{N})] \subseteq \mathcal{O}(\mathbf{E})$ by construction, so j_k is an MT-morphism by Lemma 4.3. Moreover, $j_1 \circ f = j_2 \circ f$ since $f[M] = B$ and $j_1|_B = j_2|_B$.

We claim that f is the equalizer of $j_1, j_2 : \boldsymbol{N} \to \mathbf{E}$. Let $q : \boldsymbol{Q} \to \boldsymbol{N}$ be an MT-morphism with $j_1 \circ q = j_2 \circ q$. For any $x \in Q$, we have $j_1(q(x)) = j_2(q(x))$, so $j_1(q(x)) \in j_1[N] \cap j_2[N] = j_1[B]$. Since j_1 is injective, $q(x) \in B$. Hence $q[Q] \subseteq B$.

Define $\bar{q} : \boldsymbol{Q} \to \boldsymbol{B}$ by $\bar{q}(x) = q(x)$. Since B is a complete Boolean subalgebra of N and q is a complete Boolean homomorphism, $\bar{q}$ is a complete Boolean homomorphism. For $u \in \mathcal{O}(\boldsymbol{Q})$, we have $q(u) \in \mathcal{O}(\boldsymbol{N})$. As $q(u) \in B$, we obtain $q(u) \in B \cap \mathcal{O}(\boldsymbol{N}) = \mathcal{O}(\boldsymbol{B})$. Hence $\bar{q}[\mathcal{O}(\boldsymbol{Q})] \subseteq \mathcal{O}(\boldsymbol{B})$, and Lemma 4.3 implies that $\bar{q}$ is an MT-morphism. Clearly $f \circ \bar{q} = q$, and $\bar{q}$ is unique because f is injective. Therefore f is a regular monomorphism. □

Remark 6.4. In Proposition 4.7 m is injective MT-morphism and $m[\mathcal{O}(B)] = \mathcal{O}(N) \cap B$. Thus, by Theorem 6.2 every MT-morphism f can be factorized into epi and extremal mono. Thus, by Theorem 6.3, epi and regular mono.

Proposition 6.5. *The category* MT *is not balanced. A morphism in* MT *is both an epimorphism and a monomorphism if and only if it is bijective; such a morphism is an isomorphism if and only if it is open (equivalently, $\mathcal{O}(f)$ is surjective).*

Proof. By Corollary 5.2, epimorphisms are surjective; by Lemma 6.1, monomorphisms are injective. Hence epi+mono = bijective. By Lemma 5.3, a bijective MT-morphism is an isomorphism iff $\mathcal{O}(f)$ is surjective. The map h in Proposition 7.3 is bijective but not open, hence MT is not balanced. □

7. $\mathcal{O}$ has no adjoint

7.1. **No right adjoint.** In Frm, extremal epimorphisms are strong (see, e.g., [PP12, p. 50]). This together with Theorem 5.4 and the fact ([PP12, p. 25]) that surjective frames homomorphisms are extremal epimorphisms yields:

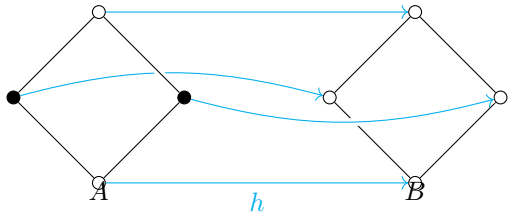
Corollary 7.1. *$\mathcal{O}$ preserves extremal, regular, epimorphisms.* **I feel like this result should be mentioned at the end of this section. Like some collection of good properties of $\mathcal{O}$.**

Proposition 7.2. *$\mathcal{O}$ preserves extremal and regular monomorphisms.*

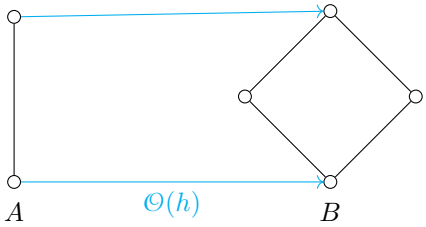
Proof. Since extremal monomorphims in MT are regular, it suffices to show regular are preserved. This follows from the fact that equalizers are preserved. □

Proposition 7.3. *$\mathcal{O}$ does not preserve epimorphisms.*

Proof. Let A be the four-element MT-algebra where only 0 and 1 are open, see Fig. 1, and let B the four-element algebra where each element is open. Then the obvious map from $h : A \to B$ is an onto MT-morphism, and hence an epimorphism.



However, $\mathcal{O}(h)$ is the unique bounded lattice map from the two-element boolean algebra to the four-element boolean algebra:



Clearly, $\mathcal{O}(h)$ is not an epimorphism. □

Theorem 7.4. *There is no right adjoint to $\mathcal{O} : \mathsf{MT} \to \mathsf{Frm}$.*

Proof. Suppose $\mathcal{O}$ has a right adjoint $\mathcal{R} : \mathsf{Frm} \to \mathsf{MT}$. Then $\mathcal{O}$ is a left adjoint, so preserves epimorphisms, contradicting Proposition 7.3. □

7.2. **No left adjoint.** The argument for the non-existence of a right adjoint relied on the fact that $\mathcal{O}$ does not preserve epimorphisms. For left adjoints, the dual property would be preservation of monomorphisms. However, $\mathcal{O}$ does preserve monomorphisms (since monomorphisms in both MT and Frm are simply injective maps), so the same strategy cannot apply. **Does $\mathcal{O}$ preserve extremal monos?** We therefore employ a different argument.

The category CBA of complete boolean algebras and complete boolean homomorphisms is a full subcategory of MT via the functor $\iota : \mathsf{CBA} \to \mathsf{MT}$ given by $\iota(B) = (B, \mathrm{id})$ and $\iota(h) = h$ for $B \in \mathsf{CBA}$ and $h \in \mathsf{CBA}(A, B)$.

Let $\mathcal{U} : \mathsf{MT} \to \mathsf{CBA}$ be the forgetful functor.

Lemma 7.5. *The forgetful functor $\mathcal{U} : \mathsf{MT} \to \mathsf{CBA}$ is left adjoint to $\iota : \mathsf{CBA} \to \mathsf{MT}$.*

Proof. For each $M \in \mathsf{MT}$ and $A \in \mathsf{CBA}$, define

$$\varphi_{M,A} : \mathsf{CBA}(\mathcal{U}(M), A) \longrightarrow \mathsf{MT}(M, \iota(A))$$

by $\varphi_{M,A}(f) = f$, i.e., the set-theoretic identity.

To see this is well-defined: if $f : \mathcal{U}(M) \to A$ is a complete boolean homomorphism, then for each $a \in M$,

$$f(\Box a) \leq f(a) = \mathrm{id}\, f(a)$$

since $\Box a \leq a$. Thus f is an MT-morphism $M \to \iota(A)$.

Conversely, every MT-morphism $M \to \iota(A)$ is a complete boolean homomorphism $U(M) \to A$. Hence $\varphi_{M,A}$ is a bijection.

Naturality in M: Let $h : M' \to M$ be an MT-morphism. We must show that for $f : \mathcal{U}(M) \to A$,

$$\varphi_{M',A}(f \circ \mathcal{U}(h)) = \varphi_{M,A}(f) \circ h.$$

Both sides equal $f \circ h$ as set maps, so this holds.

Naturality in A: Let $g : A \to A'$ be a complete boolean homomorphism. We must show that for $f : \mathcal{U}(M) \to A$,

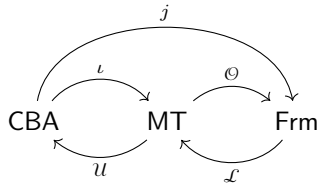
$$\varphi_{M,A'}(\iota(g) \circ f) = \iota(g) \circ \varphi_{M,A}(f).$$

Both sides equal $g \circ f$ as set maps, so this holds.

Thus φ is a natural isomorphism, proving $\mathcal{U} \dashv \iota$. □

Proposition 7.6. *There is no left adjoint to* $\mathcal{O} : \mathsf{MT} \to \mathsf{Frm}$.

Proof. Suppose $\mathcal{L} : \mathsf{Frm} \to \mathsf{MT}$ is left adjoint to $\mathcal{O}$. By Lemma 7.5, CBA is a reflective subcategory of MT with reflector $\mathcal{U}$. The composite $\mathcal{U} \circ \mathcal{L} : \mathsf{Frm} \to \mathsf{CBA}$ would then be left adjoint to the inclusion $j = \mathcal{O} \circ \iota : \mathsf{CBA} \to \mathsf{Frm}$, as shown by the following diagram of adjunctions:



But the inclusion $j : \mathsf{CBA} \to \mathsf{Frm}$ has no left adjoint: the free complete boolean algebra on a frame does not exist in general (equivalently, CBA is not reflective in Frm). This contradiction shows that $\mathcal{L}$ cannot exist. □

Theorem 7.7. *The open functor* $\mathcal{O} : \mathsf{MT} \to \mathsf{Frm}$ *has neither a left nor a right adjoint.*

Proof. No right adjoint by Theorem 7.4; no left adjoint by Proposition 7.6. □

References

[AHS06] Jiˇ rí Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories: the joy of cats*, Repr. Theory Appl. Categ. (2006), no. 17, 1–507, Reprint of the 1990 original [Wiley, New York; MR1051419]. MR 2240597

[BR23] Guram Bezhanishvili and Ranjitha Raviprakash, *McKinsey-Tarski algebras: an alternative pointfree approach to topology*, Topology Appl. **339** (2023), Paper No. 108689, 30. MR 4647809

[Fun59] N. Funayama, *Imbedding infinitely distributive lattices completely isomorphically into Boolean algebras*, Nagoya Math. J. **15** (1959), 71–81.

[Grä78] G. Grätzer, *General lattice theory*, Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, vol. Band 52, Birkhäuser Verlag, Basel-Stuttgart, 1978. MR 504338

[Joh82] P. T. Johnstone, *Stone spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, 1982. MR 698074

[Lag74] Robert Lagrange, *Amalgamation and epimorphisms in m complete Boolean algebras*, Algebra Univers. **4** (1974), 277–279 (English).

[MT44] J. C. C. McKinsey and A. Tarski, *The algebra of topology*, Ann. of Math. (2) **45** (1944), 141–191.

[Nöb54] G. Nöbeling, *Grundlagen der analytischen Topologie*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, vol. Band LXXII, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. MR 68197

[PP12] J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, Springer Basel, 2012.

[Rav25] Ranjitha Raviprakash, *McKinsey-Tarski Algebras: An Alternative Pointfree Approach to Topology*, ProQuest LLC, Ann Arbor, MI, 2025, Thesis (Ph.D.)–New Mexico State University. MR 4933004

[RS63] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, Monogr. Mat., vol. 41, Panstwowe Wydawnictwo Naukowe, Warszawa, 1963.